MANAGERIAL AND DECISION ECONOMICS, VOL. 17. 27–43 (1996)

Nonlinear Dynamical Systems and Inventory Management

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Surprisingly, deterministic time series can generate highly irregular, random-appearing trajectories. These deterministic time series result from nonlinear dynamical systems of differential or difference equations. The random appearance displayed by these systems is called *nonlinear dynamical complexity*. Properties of nonlinear complex systems include aperiodic random appearance, sensitive dependence on initial conditions and model parameters, and nonstationarity. Experiments involving the operation of simulated business environments and theoretical nonlinear dynamical models for inventory are reviewed to explore motivating factors that can give rise to demand with nonlinear complexities. The experimental and theoretical evidence reviewed indicates that nonlinear complexities in demand have significant implications for inventory management. Thus, researchers and practitioners in inventory management need to consider these properties when choosing inventory management methods. Characteristics of nonlinear dynamical systems and their implications for inventory management are presented in this paper. The use of the Brock, Dechert, and Scheinkman (1987) (BDS) test for nonlinear dependence is demonstrated on actual demand data.

INTRODUCTION

Numerous economic and financial time series exhibit evidence of *nonlinear complexities*, i.e. deterministic components that appear random using conventional time series methods (see Ashley and Patterson, 1989; Barnett and Chen, 1988; Brock and Sayers, 1988; Frank and Stengos, 1988, 1989; Scheinkman and LeBaron, 1989; Hsieh, 1989, 1991, 1993; Mignacca and Gallegati, 1994). These components result from time dependent nonlinear models known as nonlinear dynamical systems (Jackson, 1991). Furthermore, Pinder (1994) presents empirical evidence of nonlinearities in demand data. In addition to the nonlinear complexities found in field data, Sterman (1989), Mosekilde *et al.* (1991) and Paich and Sterman (1993) describe nonlinear complexities found in experiments involving the operation of simulated business environments. Boldrin (1989), Day and Walter (1989), Day (1994), Goodwin (1990), Medio (1992) and others present theoretical models of inventory systems that exhibit nonlinear complexity. Thus, nonlinear complexities have been theoretically postulated and observed in field and experimental data.

CCC 0143–6570/96/010027–17

Time series with nonlinear complexities can create difficulties for inventory management. Stochastic inventory models depend on the probabilistic properties of the demand for the firm's goods and services. Research in stochastic inventory models in summarized by Hax and Candea (1984). Lordahl and Bookbinder (1994) show that deviations from the assumption of normality increase expected annual inventory costs. More generally, Iyer and Schrage (1992), Sinha and Matta (1991), and Zheng (1992) show that departures from the independence and stationarity assumptions increase holding costs, backorder costs, and stockouts.

Developing improved forecasts for demand with nonlinear complexity also presents difficulties for inventory management. The nonlinear structure of these models produces extreme sensitivity in the demand forecasts. Small errors in either data measurement or parameter estimated lead to large forecasting errors. These, in turn, can lead to increased inventory costs as well as capacity and scheduling problems.

This paper describes the difficulties in inventory management inherent in demand data with nonlinear complexities, how to detect nonlinear complexities, and appropriate inventory models to reduce inventory holding and backorder/stockout costs when demand data contain nonlinear complexities. The properties of nonlinear dynamical complexity are given in the next section of the paper along with implications of these properties for inventory management. Previous nonlinear dynamical inventory models with nonlinear complexity are discussed in the third section. Measures for detecting nonlinear dynamical complexity are explained in the fourth section. The fifth section presents a step-by-step analysis of actual weekly demand for oil filters to demonstrate an apparently innocuous time series which contains nonlinear dependencies. This section also presents the application of several inventory policies to this data which shows that (s,S) inventory policies are not optimal for this demand data. It is also demonstrated in the fifth section that forecasting improvements suggested by tests for nonlinear complexity lead to reduced inventory costs. The final section summarizes the importance of nonlinear dynamics for inventory management and implications for further research.

NONLINEAR COMPLEXITY AND ITS IMPLICATIONS FOR INVENTORY MANAGEMENT

Nonlinear dynamical systems have several distinguishing characteristics. The most notable is that *deterministic* difference or differential equations describing the state variables of the system generate time series that appear *random*. Despite their random appearance, plots of state space behavior are bounded. Another distinguishing feature of these systems is that small changes in the initial conditions result in significantly different time series values after only a few time periods. Similarly, small changes in a coefficient, or parameter, of an equation result in significantly different time series values. Thus, these systems are extremely sensitive to measurement and initial conditions.

To illustrate these characteristics, consider the *logistic map*, a univariate nonlinear dynamical system given by:

$$X_{t+1} = \alpha X_t(1 - X_t), \qquad 0 < \alpha \leq 4 \qquad 0 < X_t \leq 1 \tag{1}$$

For $0 < \alpha < 3$, the model is 'well behaved' with stable steady-state solutions. As α increases above 3, the number of steady-state solution increases rapidly to cause complex behavior that appears random.

As a population growth model, X_t is the proportion of the population in generation t with a given characteristic. For example, the percentage of the population in the next generation with a specific trait is proportional to the current percentage with the trait times the current percentage without the trait. Applications of this model occur in epidemiology, business (Goodwin, 1990; Baumol and Benhabib, 1989), and macroeconomic models (Goodwin, 1990; Medio 1992). Goodwin (1990) presents a model of innovation capacity and investment required to create new capacity based on the logistic model. The nonlinear complexities in this model prevent the system from reaching an equilibrium capacity. Baumol and Benhabib (1989) present a logistic model for advertising. Again, nonlinear complexities prevent the system from reaching an equilibrium state.

To illustrate the apparent randomness in such models, consider the two time series in Fig. 1. Times series A is completely deterministic and was generated using Eqn. (1) with $\alpha = 3.95$ and an initial value of $X_0 = 0.05$. A Gaussian pseudorandom process generated time series B. Pseudorandom processes are actually nonlinear dynamical systems. Significant research effort has gone into the development of pseudo-random number algorithms to make then as random as possible (Press *et al.*, 1992). Data generated by the Gaussian pseudo random algorithm used here have been shown to pass tests for normality and independence.

Demand distribution parameters are used to determine the reorder point of an inventory policy. The mean serves to estimate demand during lead time and variance (or some other measure of spread Lordahl and Bookbinder, 1994) serves as a determinant of safety stock and service level. When demand distribution parameters are not stationary different inventory policy values are required for each change in the parameters. Data that are not independent are not identically distributed, i.e. not stationary. Therefore, the lack of independence and identical distribution parameters for data with nonlinear complexities causes the estimates of demand during lead time and

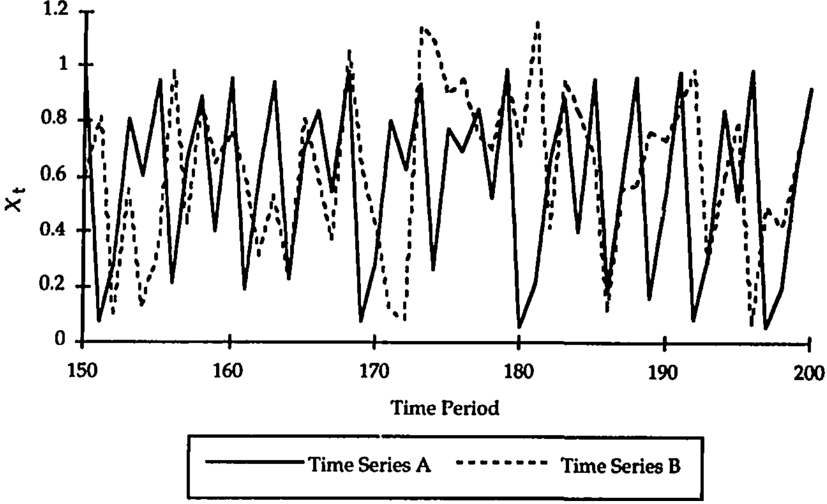


Figure 1. A random time series and a deterministic time series.

safety stock to be inaccurate. As a result, this leads to incorrect reorder points, causing increased inventory costs.

Time series generated by nonlinear dynamical processes are not stationary (Ruelle, 1989). Seasonality, backorders and stockouts can readily affect demand, creating nonlinear dependence in the time series. Sinha and Matta (1991) show that (R,S) models are optimal for stationary stochastic multi-echelon situations with finite distribution parameters. Iyer and Schrage (1992) show that (s, S) inventory policies are not optimal for demand that is not IID. They also discuss the effects of using optimal settings of (s, S) parameters in the nonoptimal situation when demand is not independent. Thus, stochastic inventory models, such as (s, Q), (s, S), (R, S), and (R, s, S), that require stationarity yield sub-optimal inventory policies for data with nonlinear complexity. Hence, it is crucial to detect whether apparently random data may have nonlinear dependencies.

Another characteristic of nonlinear dynamics is sensitivity to changes in initial conditions. In Fig. 2, the initial value changes by only 0.01%. A significant divergence of the two time paths occurs by approximately the fifteenth time period. Unlike most deterministic systems, the divergence is not readily predictable based on the magnitude of the change in the initial value. Hence, measurement of the time interval, starting time, and initial value require extreme accuracy and precision, making long-term forecasting of time series with nonlinear dynamical components difficult. Extreme dependence upon initial conditions precludes independence, and violates the requisite assumptions of stochastic inventory models.

In much the same manner that small changes in the initial value affect nonlinear dynamical systems, small changes in parameter values result in rapidly divergent time paths. Figure 3 illustrates a situation in which α of Eqn. (1) changes by only 0.01%. This sensitivity makes long-term forecasting suspect unless parameter estimates offer extraordinary accuracy.

Each variable in a system of nonlinear dynamical equations corresponds to a state space dimension. State space variables can be plotted against each other to observe the system's behavior through time. These plots are called *phase portraits*. For a univariate time series, lagged variables are used to create the phase portrait. If the state of the system is *attracted* to a particular region or set of points in the phase portrait, then that region or set of points is known as an *attractor*. Thus, an attractor is a 'picture' of the limiting behavior of the state of a system. Limiting behavior can be a fixed equilibrium points, a cycle of point, or an aperiodic bounded set of points. The last form of behavior arises from nonlinear complexities and results in aperiodic behavior that

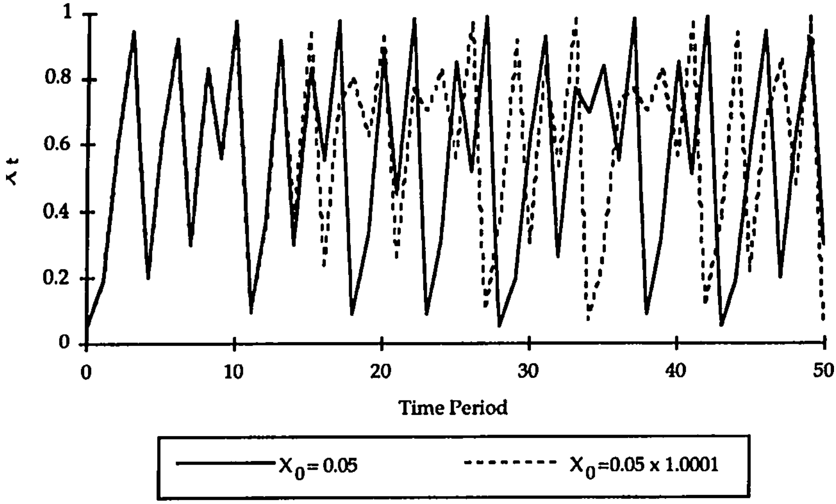


Figure 2. Sensitive dependence on initial conditions.

appears random. Grassberger and Procaccia (1983) and Ruelle (1989) provide formal mathematical definitions of attractors.

Using Eqn. (1) with $\alpha = 3.95$ and an initial condition of $X_0 = 0.05$, the phase portrait for the logistic map is generated by plotting the current state, X_t, against next period's state, X_{t+1} (see Fig. 4). The random appearance in Fig. 1 is now observed to be limited to the parabolic cluster of points in Fig. 4. This cluster of points is the attractor for the logistic map and represents the limiting behavior for this system. A significant feature of systems with nonlinear complexity is that they exhibit an attractor and do not fill the entire state space. In contrast, a bivariate random process would completely fill the two-dimensional state space. It is surprising that the regular parabolic shape in Fig. 4 could generate the apparently random behaviour in Figs. 1 through 3. To explain how this comes about, Fig. 5 traces ten sequential states in Fig. 4. The time path traced in Fig. 5 shows that while a given point indicates the state in the next period, it does not indicate subsequent states without forward iteration through each intervening state. Without an indication of future states, random appearance ensues.

Because nonlinear dynamical systems appear random, are not IID, and are extremely sensitive to initial conditions and model parameters, they describe deterministic systems with endogenous complex behavior and instability. On the other hand, the aperiodic bounded attractors of these systems can aid in describing their general limiting behavior. These properties are strikingly different than the properties of independence and stationarity assumed in stochastic models used for inventory management.

NONLINEAR DYNAMICAL MODELS OF DEMAND

Several nonlinear dynamical demand and inventory models have been postulated and examined by researchers in economics and marketing. These models present experimental and theoretical evidence of demand with nonlinear complexities and provide motivating factors for such demand.

Sterman (1989) and Mosekilde *et al.* (1991) present results from an inventory management simulation in which subjects attempt to operate a production-distribution chain to minimize costs. The simulated system contains multiple endogenous factors, feedbacks, nonlinearities, and time delays. In these experiments, the interaction of individual decisions with the structure of the simulated firm produces dynamics that systematically diverge from optimal behavior. The resulting inventory levels display nonlinear complexities

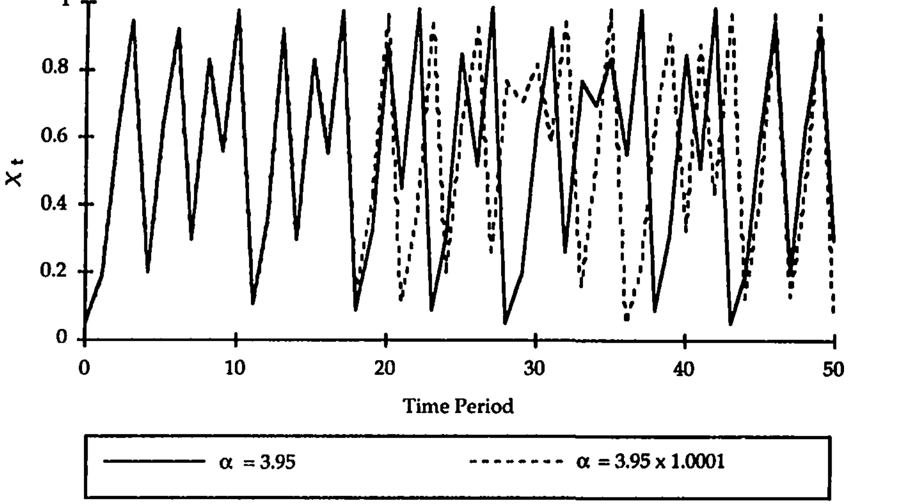


Figure 3. Sensitive dependence on model parameters.

over various parameter ranges. This implies that small changes in either the situation, such as rates of information update, order lags or lead time, or decision parameters can result in radically different behavior. In this simulation demand in not a random variable; demand is 4 units

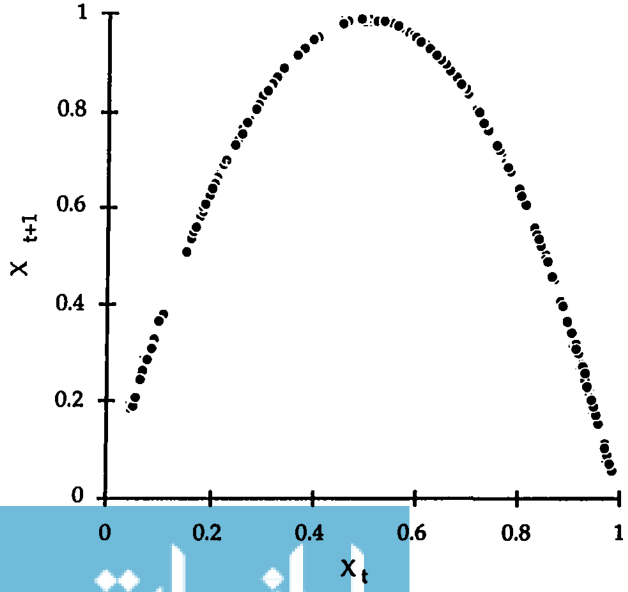


Figure 4. Attractor for the logistic map.

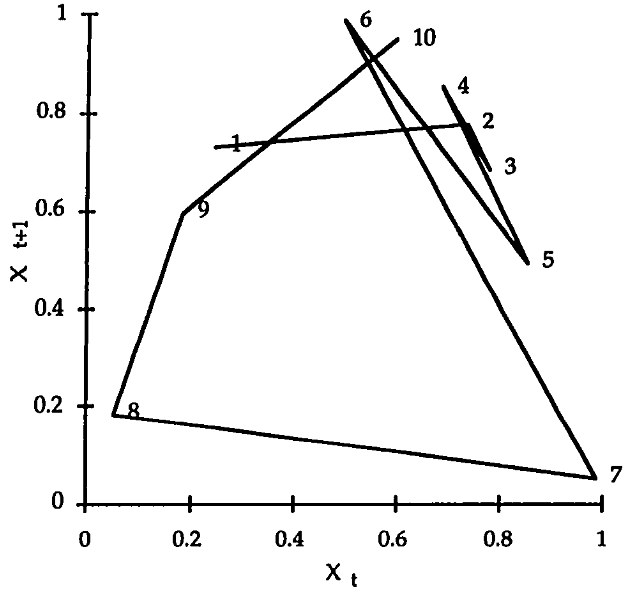


Figure 5. Interpoint connections for the logistic map.

for the four periods and then steps up to 8 units for the rest of the simulation. Nonlinear complexities in inventory levels are due to nonlinearities in stocking decision policies. Thus, even with a 'simple' pattern of demand, nonlinear dynamical complexities can occur in inventory systems due to responses by decision makers.

In a simulation of price and capacity choices for an airline, Paich and Sterman (1993) show that dynamical complexities occur in price and quantity settings due to feedback and time lag mechanisms. Furthermore, these dynamical complexities often led to extremes of either very rapid growth or bankruptcy.

Mosekilde *et al.* (1992) show that markets characterized by long lags, strong positive feedbacks, accumulations, and nonlinearities (such as commercial real estate, shipping, and capital goods) suffer more from instability and complexity than those with fewer feedback (such as soft goods and services). These lags, feedbacks, accumulations, and nonlinearities are frequently created by inventory policies. Thus, inventory policies can serve as a mechanism creating instability and nonlinear complexity.

Day (1994) postulates market mechanisms that describe the dynamic behavior of price and quantity in individual competitive markets. In one particular model, Day describes the 'market mediators' (wholesalers, retailers, merchants, brokers, or trading specialists) who supply demanders out of inventory at an announced price and restock inventories by purchasing from suppliers, again at an announced price. Thus, as in the real world, suppliers and demanders do not bargain directly with each other; they simply carry out their transactions with the market mediator. As markets attempt to clear, price and quantity are not equilibrated and there is either excess supply (accumulation of inventory) or excess demand (depletion of inventory or possibly stockout). Like Sterman (1989) and Mosekilde *et al.* (1992), Day shows that the feedbacks and time lags due to the market mediator can lead to nonlinear complexities; including the rapid growth or bankruptcy shown in Paich and Sterman's (1993) experiments. Note that the subjects in Sterman's simulations provide the same function as Day's market mediators.

Goodwin (1990) presents several nonlinear dynamical models for economics. Each of these models displays the attributes of nonlinear complexity and are based on well-known economic models. One model, developed from classical economic theory, relates the production of an agricultural good (corn) in response to a quadratic demand function and is a variation of the Logistic Map model. Goodwin (1990) also uses two cyclic models from the 1930's, the Lundberg–Metzler inventory cycle model and the Hansen–Samuelson business cycle model, to develop two models of production, demand and inventory that have nonlinear complexity. Both of these models are related to the Rössler band (Jackson, 1991) and have attractors similar to that in Fig. 6. Brock and Sayers (1988) analyze macroeconomic data for signs that business cycle effects may be attributed to nonlinear dynamical systems.

Medio (1992) uses Metzlar (1941), Gandolfo (1983), and Lorenz (1989) to derive a Rössler-type inventory business cycle model similar to Goodwin's (1991) models. Medio extends Goodwin's work by providing the proofs to show mathematically how and why these systems have nonlinear dynamical complexity. These proofs are accomplished by calculating the Liapunov exponent for these systems; a measure of sensitive dependence on initial conditions and instability of differential equation solutions.

Boldrin (1989) provides a rationale for the existence of chaotic competitive-equilibrium paths with the context of a simple, aggregated optimal-growth model with two sectors. This model has a nonlinearly complex demand component that is tied to a utility function, labor wage rates, capital stock, price, and rental for two industries. Furthermore, Boldrin provides mathematical proofs of the existence of nonlinear complexities in this model.

Day and Walter (1989) present a very long-run macroeconomic growth model in which population technology, and social infrastructure undergo regime changes. The regimes of the model are shown to compare favorably with actual regimes over several epochs. The mathematical analysis of this model shows that nonlinear complexities occur during the regime changes.

Hibbert and Wilkinson (1994) present a model of brand competition (market share) based on a predator/prey model (Kelly and Peterson, 1991). The nonlinear complexities of demand that arise are due to the competition between two consumer goods firms rather than to actions of consumers.

These models and experiments show that nonlinearly complex demand time series can arise due to many factors such as: inventory decision rules, timelags or lead time, nonlinear supply and demand functions, business cycle effects on wage and capital rates, innovations in technology or capital, or the interaction between competing firms. These models and experiments demon-

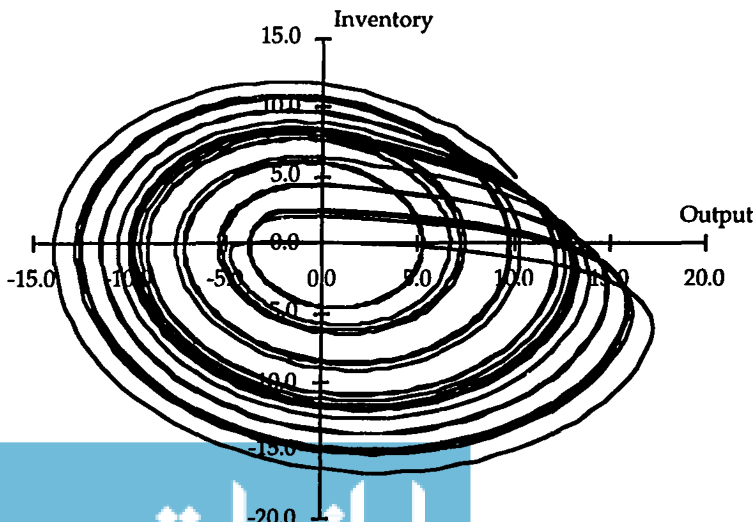


Figure 6. Attractor for the Rössler Band.

strate that nonlinear complexity can occur in demand time series and that shocks previously thought to be exogenous can be generated through the endogenous structure of demand and inventory models. Furthermore, the nonlinear complexities can result in extreme behavior results of very rapid growth or bankruptcy. Thus, the experimental and theoretical evidence indicates that nonlinear complexities in demand have significant implications for inventory management.

DETECTING NONLINEAR COMPLEXITY IN DEMAND DATA

The systems described in the previous section were derived theoretically rather than empirically. Furthermore, there are occasions when not all variables of a nonlinear dynamical system are observable to the analyst/researcher; as is the case when competitors' cost or demand information is proprietary. Thus, nonlinear dynamical research in economics and finance has concentrated on detecting chaos through the analysis of existing time series data by means of tests to distinguish between random and deterministic time series.

The key to detecting the presence of nonlinear complexity is the dimension of a system's attractor. Because the attractor of nonlinear complex system is noncontinuous, aperiodic, and bounded, the dimension of the attractor of a chaotic system is noninteger and is less than the dimension (an integer) of the state space. The dimension of the state space is the *embedding dimension*. Because a random process completely fills its state space, the dimension of its attractor is the embedding dimension (Barnett and Chen, 1988). Grassberger and Procaccia's (1983) *correlation dimension* approximates the dimension of the attractor.

Correlation Dimension

The correlation dimension is determined by first computing the *Correlation integral*. Let X_t be a vector of T observations from a time series. After choosing an appropriate embedding dimension, m, an m-dimensional vector is formed: $X_t^m = (X_t, X_{t+1}, \ldots, X_{t+m-1})$. Each m vector is of length $T-m+1$. Calculate the correlation integral:

$$C_{m,T}(\varepsilon) = \#\{(t, s), 0 < t < s < T: \|X_t^m - X_s^m < \varepsilon\} \div ((T-m+1)(T-m)/2) \quad (2)$$

where $\| \quad \|$ is the sup norm. The correlation integral measures the fraction of pairs of points (X_t^m, X_s^m) in the m-dimensional state space that are within a distance ε of each other. Thus, the correlation integral is the probability that two points chosen at random are less than ε units apart.

To calculate this probability, an embedding dimension, m, must be chosen. This is achieved by plotting the phase portrait for increasing lags of the time series. If there appears to be an attractor, then m is an appropriate embedding dimension.

As ε increases, C_m should increase proportionately to ε^D, where D is the correlation dimension. Thus:

$$C_m \propto \varepsilon^D \quad (3)$$

or

$$\ln(C_m) = D \ln(\varepsilon) + \text{constant} \quad (4)$$

D is estimated by calculating $C_{m,T}(\varepsilon)$ for increasing values of ε and regressing $\ln(C_{m,T}(\varepsilon))$ against $\ln(\varepsilon)$. Convergence will occur when m is higher than D since an attractor embedded in a higher dimension retains its true dimension because of the correlation between points (Brock, 1986; Brock and Dechert, 1988). High values of D are evidence for random systems; low values of D indicate deterministic systems.

BDS Test Statistic for Independence

Because there is no statistical inference for the correlation integral, Brock *et al.* (1987) developed a test statistic (the BDS test) based on the correlation integral. The null hypothesis for this test is that the observations in the time series X_t are IID.

If X_t is a random sample of IID observations, then:

$$C_m(\varepsilon) = C_1(\varepsilon)^m \quad (5)$$

and $C_{m,T}(\varepsilon)$ and $C_{1,T}(\varepsilon)^m$ are estimated for a sample of size T by Eqn. (2). Brock *et al.* (1987) show that the BDS statistic:

$$W_{m,T}(\varepsilon) = \sqrt{T}\left(C_{m,T}(\varepsilon) - C_{1,T}(\varepsilon)^m\right)/\sigma_{m,T}(\varepsilon) \tag{6}$$

converges in distribution to a standard normal, N(0,1), as T increases. In Eqn. (6), $\sigma_{m,T}^2(\varepsilon)$ is an estimate of the asymptotic variance of $(C_{m,T}(\varepsilon) - C_{1,T}(\varepsilon)^m)$. Thus, to test for independence and randomness of the observations in a time series, compute the BDS statistic. If the absolute value of the test statistic is large, the null hypothesis of IID (randomness) will be rejected. Conversely, if the data are IID then a small BDS statistic indicates stationarity and independence.

Furthermore, the BDS statistic can be applied to the residuals of a forecasting model to determine if there is only white noise remaining in the residuals. If there is some structure remaining in the residuals (they are not IID) then additional structure is required in the model. The additional structure can take a wide variety of forms, including: nonlinear model structure, additional variables, autoregressive conditional heteroscedasticity (ARCH, see Engle, 1982), or simultaneous equations.

Calculation of the BDS test provides three important pieces of information regarding the reliability of a demand forecast. First, a large BDS statistic indicates that the time series may rapidly diverge from slightly different estimates of the initial conditions. Thus, the initial conditions must be measured as precisely as possible to forecast demand accurately. Second, a large BDS statistic limits the time horizon for forecasting accuracy. Unfortunately, there is no direct measure to relate the BDS statistic to the length of time before the divergence in time paths is significant. Third, a large BDS statistic indicates that estimates of structural model coefficients must have extremely tight confidence intervals.

Researchers have successfully used the BDS test to detect nonlinearities in economic and financial data. Scheinkman and LeBaron (1989) detect significant departures from random-walk behavior in returns on the equal-weighted and value-weighted NYSE stock indices for both weekly and monthly data. Similarly, Hsieh (1991) shows that 15-minute stock returns exhibit nonlinear dependence. Hsieh (1989) finds significant departures from random normal (Gaussian) models for forecasted exchange rates of the US dollar against the Japanese yen, the British pound, and the German mark. Again using currency data, Hsieh (1993) shows that daily log price changes in currency future contracts are not IID. Mignacca and Gallegati (1994) use the BDS test on residuals of an ARMA model of US GNP to identify different regimes in the data.

Pinder (1994) presents empirical evidence of nonlinearities in demand data. This research shows that the BDS test identifies departures from IID assumptions not detected by the Durbin–Watson test for autocorrelation, the Q^* test (Kmenta, 1986) for autocorrelation, or Engle's ARCH test (Engle, 1982; Kmenta, 1986) for autoregressive conditional heteroscedasticity (ARCH). Thus, the BDS test has been used successfully in several contexts to detect dependencies that are undetected by other statistical measures.

NONLINEAR DEPENDENCE IN AN ACTUAL DEMAND TIME SERIES

The following analysis of an actual demand time series demonstrates two applications of the BDS test and shows the effects of applying stochastic inventory policies to demand data with nonlinear dependencies. First, the use of the BDS statistic as a test for conformance to the IID assumptions of stochastic inventory models is demonstrated. Second, the BDS test is featured as an indication of forecasting process impovement. If the BDS statistic indicates that the errors are IID, then no further progress in the forecasting model development is possible. Conversely, rejection of the IID null indicates the presence of other factors associated with the data. Finally, several inventory policies are applied to the same actual demand time series to show that stochastic inventory models are not optimal for demand data with nonlinear dependencies and that forecasting improvements indicated by the BDS test reduce inventory costs.

Figure 7 shows nearly four years of weekly demand for a component of a piece of agricultural equipment. These components supply demand of both newly manufactured equipment requiring new components and for replacement components. The seasonality typically associated with agricultural equipment is readily observable. Many nonlinear complex systems contain cyclical, or oscillatory components, such as seasonality (Brock and Malliaris, 1989; Jackson, 1991).

The original data were deseasonalized by weekly multiplicative indices. Figure 8 shows the time series plot of the deseasonalized demand and reveals a downward trend in the data. This downward trend corresponds with the macroeconomic conditions in the agricultural sector of the US economy at that time.

TESTING STOCHASTIC MODEL ASSUMPTIONS

Bock, Hsieh, and LeBaron (1991) (BHL) use simulation to determine the power of the BDS statistic and provide quantiles for the BDS statistic for samples of 100, 250, and 500 observations; embedding dimensions of 2, 3, 4, and 5; and ε equal to 0.5σ and σ; where σ is the standard deviation of the time series. The critical value reported by BHL for a sample size of 100, $m = 2$, $\varepsilon = \sigma$, and a significance level of 0.05, is 2.22; at a significance level of 0.01, the critical value is 3.40.

Figure 9 presents the distribution and descriptive statistics for the deseasonalized demand. The data appear random and approximately normal. For the analysis of the 189 observations of the oil filter demand time series, the quantiles for a sample size of 100 are used to provide a conservative estimate of the distribution of the BDS statistic. An embedding dimension of 2 and an $\varepsilon = 174.2$ ($\varepsilon/\sigma = 1$) yields a BDS statistic for the deseasonalized time series of 25.24. Therefore the null hypothesis of IID can be rejected. Thus, stochastic inventory models that assume independence and stationarity of demand parameters are not appropriate for this time series.

These results are consistent with the appearance of the downward trend in Figure 8. This demonstrates the ability of the BDS statistic to assess conformance of the data to the assumptions of stochastic inventory models.

Forecasting Model Development

The magnitude of the BDS statistic for the deseasonalized demand data indicates that there are other deterministic components, such as the trend component apparent in Figure 8. At this stage of

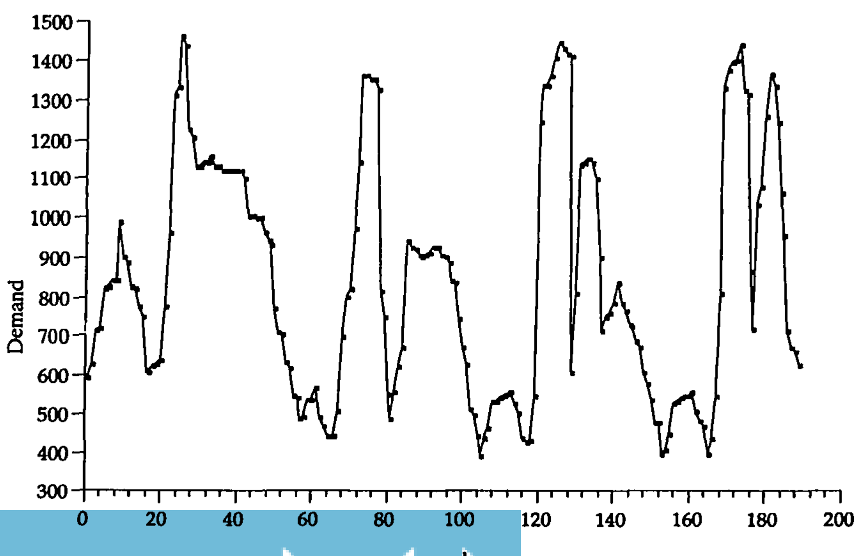


Figure 7. Four years of weekly demand.

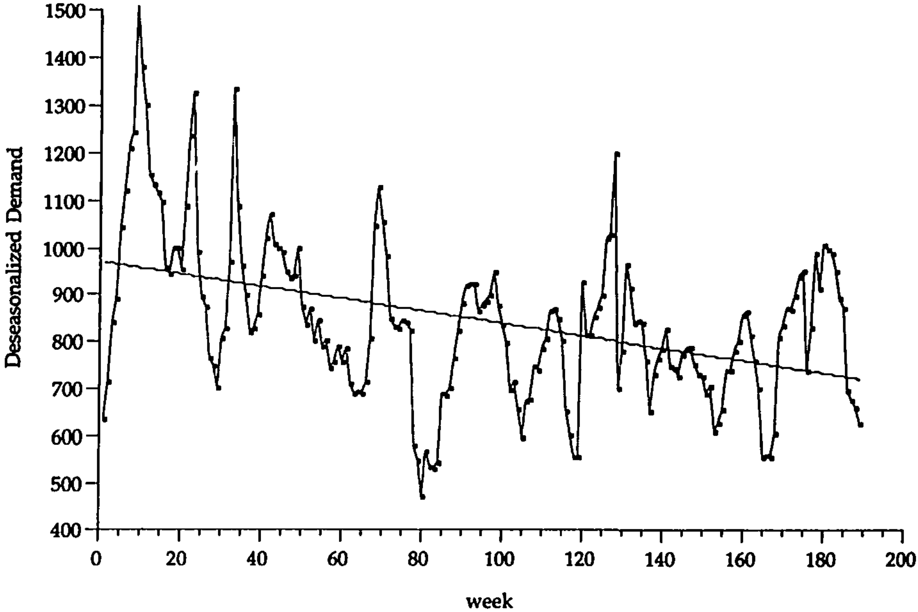
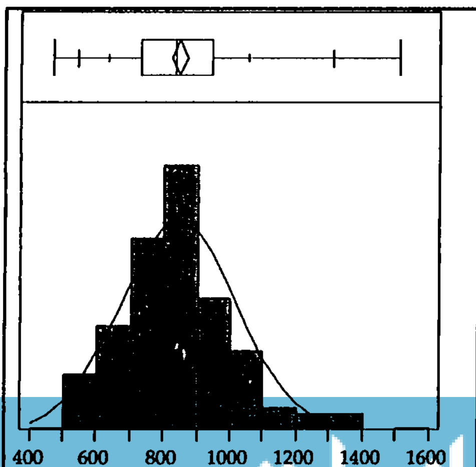


Figure 8. Deseasonalized weekly demand.

the analysis, the BDS statistic becomes an indicator of improvement in the forecasting model. Residuals of a forecasting model should be IID to insure that sufficient explanatory factors have been taken into account.

Often the first explanatory factor to be considered is the order of integration. To determine whether the series should be modeled in differences (see Figure 10) or levels, the Augmented Dickey–Fuller (ADF) (Dickey and Fuller, 1981; Engle and Granger, 1987) is used to determine the order of integration (I(d)). The null hypothe-



Quantiles		
maximum	100.0%	1510.6
quartile	75.0%	945.6
median	50.0%	835.3
quartile	25.0%	734.2
minimum	0.0%	473.0

Moments	
Mean	846.693
Std Dev	174.194
Std Err Mean	12.671
upper 95% Mean	871.688
lower 95% Mean	821.698
N	189.000

Figure 9. Distribution of deseasonalized weekly demand.

sis of the ADF test is that the series is I(1) versus a specified alternate hypothesis that the series is I(0). Engle and Granger (1987) generated statistics for a t test of this hypothesis. At a significance level of 0.05 and a sample size of 100, the critical value given by Engle and Granger (1987) is 3.17. Conducting this test on the deseasonalized data yields a t-statistic of -0.8405; this supports the null hypothesis of an I(1) series and suggests the demand data be modeled in levels.

Based on the results of the ADF, autoregressive models were investigated to describe the trend component. Testing lag lengths 1 through 12 showed the first lag to be the only significant lag. Based on these test, the oil filter demand data was estimated as an AR(1) model. The results of this regression are presented in Table 1.

The residuals of the AR(1) model in Table 1 were tested for compliance with IID assumptions. Computing the BDS statistic for the residuals, using an embedding dimension of 2 and an $\varepsilon =$ 93.8250 ($\varepsilon/\sigma = 1$), results in a test statistic of 3.18. This is significant at the 5% level. This result could be due to either nonlinear dependence or autoregressive conditional heteroscedasticity (ARCH) (Hsieh, 1991). Engle's ARCH test (Engle, 1982; Kmenta, 1986) was conducted to determine if the significance of this BDS test is due to heteroscedasticity of the residuals. The ARCH test (χ^2 value of 2.952 is not significant at the 5% level. These statistics imply that there is a nonlinear dynamical component present in the residuals of the AR(1) model.

Because of the significance of the BDS test, the first differences where tested with the BDS test. Using $m = 2$, $\varepsilon = 95.9$ ($\varepsilon/\sigma = 1$) yielded a BDS statistic of 3.04; significant at the 5% level. In contrast to the ADF test, the BDS test suggests that further information can be gained by including first difference terms.

Based on this result, as second regression model was estimated. This model extends the AR(1) model by including the lagged first difference. The results of the regression are presented in Table 2. The first difference term is not significant at the 10% level. Nevertheless, the residuals of this model were tested for IID properties. Using an embedding dimension of 2 and an $\varepsilon =$ 93.6488 ($\varepsilon/\sigma = 1$) results in a BDS statistic of 1.94. This value is not significant at the 5% level, implying that the residuals of this model are IID.

It is reasonable that the inclusion of the first difference term results in IID residuals. The significance level of the first difference term is not

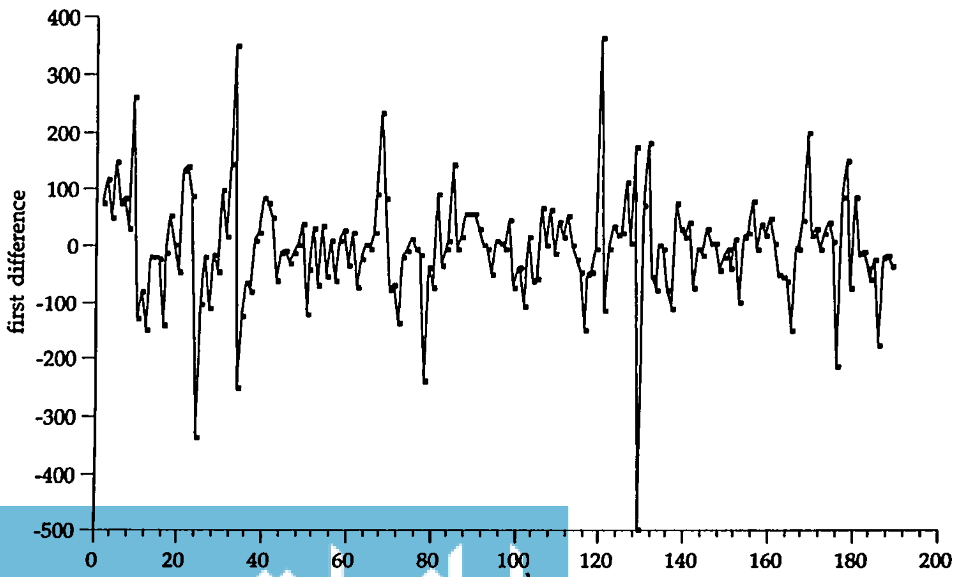


Figure 10. First differences of deseasonalized weekly demand.

Table 1 Regression Results of AR(1) Model for Deseasonalized Demand

Summary of fit					
r^2				0.711825	
Standard error of residuals				93.82498	
Mean of Response				834.9133	
Observations				188	
Analysis of variance					
Source	DF	Sum of squares	Mean square	F ratio	Prob > F
Model	1	4023177.2	4023177	457.0169	0.0000
Error	186	1637381.5	8803		
C Total	187	5660558.7			
Parameter estimates					
Term		Estimate	Std error	t ratio	Prob > |t|
Intercept		132.80584	34.1385	3.89	0.0001
Des. Demand_{t-1}		0.84331	0.03945	21.38	0.0000

so extreme as to completely preclude the term's relevance; particularly given that the ADF test was unable to detect this component.

The regression parameters in Table 2 was used to forecast deseasonalized demand. These values were then multiplied by the appropriate weekly indices to get a seasonalized forcast of demand. To measure the accuracy of the model, the actual demand was regressed against this forecast. Figure 11 demonstrates the overall fit of the model versus the actual demand. The 45° line on the plot represents the indeal fit. This regression yields an r^2 of 0.9037. As before, the residuals of this model were tested for IID properties. Using an embedding dimension of 2 and an $\varepsilon = 91.080$ ($\varepsilon/\sigma = 1$) returns a BDS statistic of 1.87. As expected, this is not significant at the 5% level. This indicates that these residuals are IID and there is little possibility of further forecasting improvement for these data.

Application of Inventory Models

Several inventory policies were applied to the oil filter demand data to demonstrate that stochastic models are not optimal for demand with nonlinear dependence and that improved forcasts reduce the average inventory costs per period. The following costs were used to determine the average cost per period of the inventory policies: setup = \$297, holding = \$15/unit/period, shortage = \$5/unit/period, lead time = 2 weeks.

Table 2 Regression Results of Modified AR(1) Model for Deseasonalized Demand

Summary of fit					
r^2				0.714066	
Standard error of residuals				93.64883	
Mean of response				848.5004	
Observations				187	
Analysis of variance					
Source	DF	Sum of squares	Mean square	F ratio	Prob > F
Model	2	4029910.9	2014955	229.7528	0.0000
Error	184	1613699.1	8770		
C Total	186	5643610.0			
Parameter estimates					
Term		Estimate	Std error	t ratio	Prob > |t|
Intercept		146.2676	35.6144	4.11	0.0001
Des. Demand_{t-1}		0.8271353	0.04117	20.09	0.0000
ΔDes. Demand_{t-1}		0.1143178	0.07321	1.56	0.1201

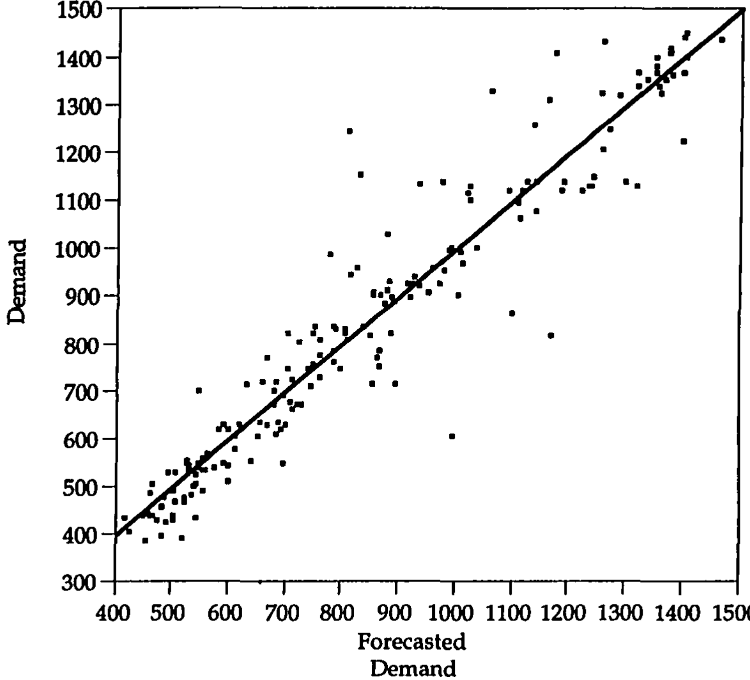


Figure 11. Plot of forecasted demand versus actual demand.

First, the Wagner–Whiten (1958) algorithm was applied to the most recent 50 periods (approximately one year) of the oil filter demand data to determine the optimal inventory policy. This policy yielded an average cost per period of $258.05. This average cost per period provides a basis of comparison for the other inventory policies and is required to establish the value of perfect information.

To determine the potential effectiveness of an (s, S) inventory policy for this demand data, an iterative search method provided the (s, S) inventory policy with the minimum average cost per period for the last 50 periods of demand. The resultant average cost per period was $484.14 (87.61% increase). This shows that even with a perfect forecast, an (s, S) inventory policy is not optimal for these data.

To determine the actual effectiveness of an (s, S) inventory policy for this demand data, two (s, S) inventory models were calculated based on the first 139 periods (nearly 3 years) of demand. The first (s, S) policy was calculated by the simultaneous determination method of Peterson and Silver (1979), which assumes stationarity, independence, and normality. The second (s, S) policy was calculated by the order-statistic method of Lordahl and Bookbinder (1994). Their method does not assume normality, but does assume stationarity and independence. These two policies were applied to the final 50 periods of actual demand. The Peterson and Silver policy yielded an average cost per period of $781.19 (202.73% increase) and the Lordahl and Bookbinder policy an average cost per period of $667.01 (158.48% increase). Because these methods do not use the last 50 periods of demand, forecasting improvements cannot reduce these costs.

To determine the effects of improved forecasting on inventory costs for these demand data, the first 139 periods of demand were used to determine three forecasts of the last 50 periods of demand. The first forecast uses the average demand of the first three years and the weekly seasonalized indices from these three years. The second forecast was calculated from an AR(1) model; structurally identical to the model shown in Table 1. The third forecast (denoted as $AR(1)^+$) was calculated from an AR(1) model augmented by the lagged first difference term; structurally identical to the model shown in Table 2. The Wagner–Whiten algorithm was applied to each forecast to determine an inventory policy for the last 50 periods. These inventory policies were then subjected to the actual demand and the average inventory costs per period determined. This represents the average cost of ordering to satisfy forecasted demand and then being subject to the actual demand. The seasonal index forecast yielded average inventory costs per period of $548.38 (112.51% increase); the AR(1) forecast yielded average inventory costs per period of $304.86 (18.14% increase); and the $AR(1)^+$ forecast yielded average inventory costs per period of $298.04 (15.50% increase).

Fixed period requirements policies of 1 to 20 periods were calculated using the last 50 periods of actual demand. The two-period policy provided the lowest average cost per period: $265.24; an increased cost of 2.79% over the optimal cost. This is the cost of a perfect four-period (two-period lead time plus two periods of demand) forcast rather than a perfect full-year forecast. Thus, if it were possible to perfectly forecast the oil filter demand a month in advance, the increase in inventory costs would only be 2.79% above the optimal inventory costs.

Next, two period fixed period requirements policies based on the three forecasts were applied to the last 50 periods of actual demand. The seasonal index forecast yielded average inventory costs per period of $556.72 (115.74% increase); the AR(1) forecast yielded average inventory costs per period of $311.93 (20.88% increase); and the $AR(1)^+$ forecast yielded average inventory costs per period of $305.86 (18.14% increase).

Table 3 presents a summary of the average cost per period of each of the inventory policies described above. These results show that the forecasting improvements provide significant reduction in inventory costs over the (s, S) inventory model costs. Comparing the costs of the (s, S) inventory policies to the costs of the Wagner–Whiten and fixed period requirements policies shows that applying (s, S) inventory models to this data results in significantly higher inventory costs. Even the relatively naive forecast using seasonal indices (a nonlinear component) provides lower inventory costs than the lowest cost (s, S) policy (Lordahl and Bookbinder, 1994) that could be derived without a forecast.

CONCLUSION

The characteristics of nonlinear dynamical systems and their implications for inventory management are presented in this paper. Systems with nonlinear complexity appear random and are extremely sensitive to initial conditions and model parameters. The experimental and theoretical evidence reviewed in this paper indicates that nonlinear complexities in demand have significant implications for inventory management. The use of the correlation dimension and the BDS test for the analysis of demand is also demonstrated. These measures distinguish stochastic from nonlinear complex (deterministic) time series.

Table 3 Summary of Inventory Policy Costs Based on Demand Forecasts

Forecast	W – W(opt)	FPR(2)	Best(s, S)	P&S(s, S)	L&B(s, S)
Actual	$258.05 0.00%	$265.24 2.79%	$484.14 87.61%	NA	NA
None	NA	NA	NA	$781.19 202.73%	$667.01 158.48%
Seas. ind.	$548.38 112.51%	$556.72 115.74%	NA	"	"
AR(1)	$304.86 18.14%	$311.92 20.88%	NA	"	"
$AR(1)^+$	$298.04 15.50%	$304.86 18.14%	NA	"	"

Note: $AR(1)^+$ denotes the AR(1) model with the lagged first difference term.

This paper shows that demand forecasting for inventory management is difficult when the demand time series contain nonlinear complexities. The correlation dimension and BDS test help to indentify further structure in forecasting models. In addition, the BDS test can identify departures from stochastic inventory model assumptions of independence and stationarity. The numerical example demonstrates that applying stochastic inventory models to data containing nonlinear complexities results in significantly higher inventory costs.

Nonlinear dynamical theory does not suggest that stochastic, econometric, and statistical methods are inappropriate. Instead, it suggests that stochastic, econometric, and statistical methods should be augmented by the analytic methods described in this paper.

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